W2.a). If $a,b,c \in \mathbb{R}_+^*$ such that $abc = 1$, then prove the following inequality:

$$\frac{1}{a^3(F_n b + F_{n+1}c)} + \frac{1}{b^3(F_n c + F_{n+1}a)} + \frac{1}{c^3(F_n a + F_{n+1}b)} \ge \frac{3}{F_{n+2}}$$

b). If $a,b,c \in \mathbb{R}_+^*$ such that $ab + bc + ca = 3$, then prove the following inequality:

$$\frac{1}{a^3(F_n b + F_{n+1}c)} + \frac{1}{b^3(F_n c + F_{n+1}a)} + \frac{1}{c^3(F_n a + F_{n+1}b)} \ge \frac{3}{F_{n+2}}$$

c) If $a,b,c \in \mathbb{R}_+^*$ such that $a + b + c = 3$, then prove the following inequality:

$$\frac{1}{a^3(F_n b + F_{n+1}c)} + \frac{1}{b^3(F_n c + F_{n+1}a)} + \frac{1}{c^3(F_n a + F_{n+1}b)} \ge \frac{3}{F_{n+2}}$$

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Solution by Arkady Alt, San Jose, California, USA.

First we note that for any $x,y,z > 0$ holds inequality

(**1**) $$\sum_{cyc} \frac{x^2}{F_n z + F_{n+1}y} \ge \frac{x+y+z}{F_{n+2}}.$$

Indeed, by Cauchy Inequality $\sum_{cyc} \frac{x^2}{F_n z + F_{n+1}y} \ge \frac{(x+y+z)^2}{\sum_{cyc}(F_n z + F_{n+1}y)} =$

$$\frac{(x+y+z)^2}{F_n \sum_{cyc} z + F_{n+1}\sum y} = \frac{(x+y+z)^2}{(F_n + F_{n+1})(x+y+z)} = \frac{x+y+z}{F_{n+2}}.$$

a) Let $(x,y,z) := \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$. Then $xyz = 1$ and $\sum_{cyc} \frac{1}{a^3(F_n b + F_{n+1}c)} =$

$\sum_{cyc} \frac{x^3yz}{F_n z + F_{n+1}y} = \sum_{cyc} \frac{x^2}{F_n z + F_{n+1}y}$. Since by AM-GM inequality $x + y + z \ge 3\sqrt[3]{xyz} = 3$

then $\sum_{cyc} \frac{1}{a^3(F_n b + F_{n+1}c)} = \sum_{cyc} \frac{x^2}{F_n z + F_{n+1}y} \ge \frac{3}{F_{n+2}}$.

b) Let $(x,y,z) := \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$. Then $ab + bc + ca = 3 \Leftrightarrow x + y + z = 3xyz$ and using

inequality (**1**) we obtain $\sum_{cyc} \frac{1}{a^3(F_n b + F_{n+1}c)} = \sum_{cyc} \frac{x^3yz}{F_n z + F_{n+1}y} = xyz\sum_{cyc} \frac{x^2}{F_n z + F_{n+1}y} \ge$

$\frac{xyz(x+y+z)}{F_{n+2}} = \frac{3x^2y^2z^2}{F_{n+2}}$. Since by AM-GM inequality $3xyz = x + y + z \ge 3\sqrt[3]{xyz}$ then

$xyz \ge 1$ and, therefore, $\sum_{cyc} \frac{1}{a^3(F_n b + F_{n+1}c)} \ge \frac{3(xyz)^2}{F_{n+2}} = \frac{3}{F_{n+2}}$.

c) Let $(x,y,z) := \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$. Then $a + b + c = 3 \Leftrightarrow xy + yz + zx = 3xyz$ and using

inequality (**1**) we obtain $\sum_{cyc} \frac{1}{a^3(F_n b + F_{n+1}c)} = \sum_{cyc} \frac{x^3yz}{F_n z + F_{n+1}y} = xyz\sum_{cyc} \frac{x^2}{F_n z + F_{n+1}y} \ge$

$\frac{(x+y+z)xyz}{F_{n+2}}$. Since by AM-GM inequality $3xyz = xy + yz + zx \ge 3\sqrt[3]{x^2y^2z^2} \Leftrightarrow xyz \ge 1$

then $x + y + z \ge 3\sqrt[3]{xyz} = 3$ and, therefore, $\frac{(x+y+z)xyz}{F_{n+2}} \ge \frac{3}{F_{n+2}}$.